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Onset of convection in a rapidly rotating fluid sphere

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The constraint imposed by rapid rotation stabilizes convective motions in a plane liquid layer, so that the critical Rayleigh number which marks their onset increases as the two-thirds power of the Taylor number. Here we formulate an asymptotic approximation to the equations governing an analogous problem in a sphere. The solutions we obtain for the asymptotic equations predict the same power law for the dependence of the corresponding critical Rayleigh number on the Taylor number. The analysis is limited at the outset to the case of axially symmetric motions, and the present result, taken in conjunction with earlier results for slow rotation, essentially completes the treatment of symmetric, convective modes of instability.

Introduction

In this paper we are interested in solving a thermal instability problem in a rotating sphere with potentially unstable temperature gradient. Our interest will be centred on the case of large Taylor number ($T > 10^{12}$), for which the combination of Galerkin's method and expansion techniques fails to provide a reliable solution to the problem (see e.g. Bisshopp 1958). This is the case of interest in geophysical and astrophysical situations, where the Taylor number is often very large indeed (e.g. $\sim 10^{27}$ in the Earth's core), but our result is limited by the assumptions of axial symmetry and the principle of the exchange of stabilities. Since explicit use of the largeness of the Taylor number (and the critical Rayleigh number) will be made throughout the analysis, this can be called an asymptotic treatment of the thermal instability problem.

In an earlier paper (Niiler & Bisshopp 1965) we have discussed the onset of convection in a rotating layer of fluid, heated from below and subject to a uniform gravitational field. There an exact solution of the stability problem can be found, and its asymptotic expansion computed. It is found that the incipient motion in the rotating layer is characterized by two distinct types of behaviour. In the region of fluid removed from rigid boundaries the gradients of the components of perturbations of the steady conductive solution are all constrained to be perpendicular to the axis of rotation in the limit where $T \to \infty$. Near the boundaries of the system there is a viscous boundary layer whose thickness approaches zero in the same limit and within which the direction of the gradients changes from one normal to the axis of rotation to normal to the boundary. The dominant terms in the equations of motion in such layers are those derived from Coriolis forces and from viscous stresses, and we shall follow current practice in calling them Ekman layers.

Here we propose to develop a solution of the thermal instability problem for a self-gravitating, rotating liquid with a uniformly distributed heat source within. A tractable problem results if we treat density variation due to change in temperature by the Boussinesq approximation, where the variation of density is taken into account only in so far as it provides buoyancy in the presence of gravity and where density and temperature changes are proportional. No effort has been made to develop an asymptotic expansion of the solution, i.e. to prove that the solution is indeed asymptotic to an exact solution. Instead, we have developed an asymptotic approximation, and we shall have to be satisfied with knowing the order of magnitude of the error which arises at each step of the analysis.

2. Equations of the problem

For the present purpose, we shall refer to Chandrasekhar's formulation (1961, ch. 6) of the equations governing marginal stability of axially symmetric perturbations of the temperature and velocity fields of a self-gravitating, rotating, fluid sphere with a uniformly distributed heat source. The perturbed velocity field is a superposition of a polodial and torodial component and can be represented in terms of two azimuth-independent scalars U' and V'. In terms of spherical co-ordinates $(r', \mu' = \cos\theta, \phi')$ we can write the perturbation velocity $\mathbf{u}'$ as

$$\mathbf{u}' = \frac{\partial[(1-\mu'^2)\,U']\,\hat{\mathbf{r}}'}{\partial\mu'} - (1-\mu'^2)^{\frac{1}{2}}\frac{1}{r'}\frac{\partial[r'^2U']\,\hat{\boldsymbol{\theta}}'}{\partial r'} + r'(1-\mu'^2)^{\frac{1}{2}}\,V'\hat{\boldsymbol{\phi}}. \tag{2.1}$$

The corresponding non-dimensional scalars, U, V (see e.g. Bisshopp 1958) satisfy the following equations in non-dimensional variables, r, μ:

$$\Delta_5^2 U - T\,\partial V/\partial z = R\,\partial\theta/r\,\partial\mu, \tag{2.2}$$

$$\Delta_5 V + \partial U/\partial z = 0, \tag{2.3}$$

$$\Delta_5(\partial\theta/r\,\partial\mu) = \partial^2([1-\mu^2]\,U)/\partial\mu^2, \tag{2.4}$$

where θ is the perturbation temperature distribution; T and R are the Taylor and Rayleigh numbers, defined respectively as $T = 4\Omega^2R_0^4/\nu^2$ and $R = 2\beta\gamma R_0^6/\kappa\nu$, R_0 is the radius of the sphere, Ω is the angular velocity of the sphere, ν is the kinematic viscosity, κ is the coefficient of thermal conductivity, ϵ is the rate at which the temperature would rise in the absence of conduction, $\beta \equiv \epsilon/6\kappa$, $\gamma \equiv 4\pi G\alpha\rho/3$, G is the gravitation constant, α is the coefficient of volume expansion and ρ is density. The five-dimensional Laplacian operator is

$$\Delta_5 = \frac{1}{r^2}\frac{\partial^2(r^2)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\{(1-\mu^2)\}}{\partial\mu^2}, \tag{2.5}$$

and $\partial/\partial z$ is the operator

$$\frac{\partial}{\partial z} = \mu\frac{\partial}{\partial r} + \left(\frac{1-\mu^2}{r}\right)\frac{\partial}{\partial\mu}. \tag{2.6}$$

In the derivation of equations (2.2)–(2.4) it has been assumed that the growth rate of a marginally stable disturbance is zero, rather than a pure imaginary number. This assumption is usually referred to as the assumption of the principle of the exchange of stabilities or as a restriction of the treatment to convective modes of instability. It can be shown (Chandrasekhar 1961) for the plane layer confined between free boundaries that the convective modes of instability are unstable at a lower Rayleigh number than the alternate, overstable modes, provided the Prandtl number (ν/κ) is sufficiently large ($> 0{\cdot}677$). Clearly, we may expect a similar result in the problem at hand, but we shall not pursue the matter in this paper.

The boundary conditions for a free boundary at $r = 1$ are that the normal component of velocity, the tangential components of viscous stress and the temperature perturbation vanish, i.e. that

$$U = 0, \quad \partial^2(rU)/\partial r^2 = 0, \quad \partial V/\partial r = 0, \quad \partial\theta/\partial\mu = 0. \tag{2.7}$$

For a rigid boundary they are that the velocity and temperature perturbation vanish, i.e. that

$$U = 0, \quad \partial U/\partial r = 0, \quad V = 0, \quad \partial\theta/\partial\mu = 0. \tag{2.8}$$

In this problem T can be considered as a number specified in advance, and then equations (2.2)–(2.4) together with boundary conditions (2.7) or (2.8) provide a characteristic-value problem for R. Because the differential equations are not separable for arbitrary values of T, it is highly unlikely that an exact solution to the full problem can be exhibited. In a previous paper (Bisshopp 1958), the problem has been solved approximately by constructing a set of functions which satisfy the boundary conditions and some of the differential equations exactly; a variational principle then provides that the remaining equations shall be satisfied in an approximate manner. This method gives a characteristic equation of an infinite degree for the Rayleigh number in terms of an expression of infinite degree in the Taylor number. From this expression numerically converging values of the critical Rayleigh number can be calculated for $T < 10^8$. The method of calculation is cumbersome for $T \sim 10^6$ and the accuracy is difficult to estimate, but nevertheless it is strongly suggested that R_c is a monotonic function of T. We are interested in the case of large, but finite $T\,(T > 10^{12})$ and want to compute the asymptotic limit of this function. Our solution taken together with the earlier calculations should provide the critical Rayleigh number for the entire range of Taylor number $0 \leqslant T < \infty$.

In an asymptotic analysis we shall adopt a somewhat different viewpoint from that which is employed when constructing an approximate solution by the variational methods. We begin by constructing an asymptotic set of differential equations, the solution of which will be the asymptotic eigenfunctions. To determine the asymptotic characteristic equation, these eigenfunctions are substituted into the boundary conditions, and these equations have to be satisfied to at least the same degree of approximation as the asymptotic differential equations. Since it is difficult to prove that such a solution is indeed asymptotic to a solution of the full set of equations and boundary conditions, we claim only to be able to estimate the order of magnitude of the errors which are made at every step of the analysis. In the present problem the analysis will receive its impetus from physical conditions which are expected to prevail in the interior of the sphere.

3. Asymptotic theory of thermal instability in a rotating sphere

In a previous paper (Niiler & Bisshopp 1965) on thermal instability of a liquid layer, heated from below, it was shown that increasing Taylor number forces the perturbed motion in the interior of fluid cells to have gradients along the axis of rotation which are of $O(T^{-\frac{1}{6}})$ compared with those normal to the axis. It was also shown that large Taylor number radically reduces the effect of viscous stress introduced at the boundaries everywhere outside a vanishingly narrow layer close to the rigid walls, so that two distinct limits of the perturbation equations are necessary. One limit governs the motion in the interior of fluid cells, while the other prevails in the boundary layer. We shall now show that similar constraints are imposed on the motion in the sphere, and again there are two limits to the perturbation equations.

In a sphere, the cylindrical co-ordinate normal to the axis of rotation is

$$\omega = (1-\mu^2)^{\frac{1}{2}} r, \tag{3.1}$$

and the co-ordinate in the direction of rotation is

$$z = \mu r. \tag{3.2}$$

Since we expect considerably larger gradients of the perturbations in the direction normal to the axis of rotation in the interior of the sphere (when $T \to \infty$), we shall introduce the transformation

$$\omega^* = T^m(\omega - \omega_0) \quad (m \geqslant 0), \tag{3.3}$$

and seek a solution of the resulting asymptotic equations which is a function of ω^* and z. It should be noted that we do not intend to take over the result $m = \frac{1}{6}$ directly from the analysis of the plane layer. We shall take over only the fact that a co-ordinate stretch is expedient, and equation (3.3) represents the general case when axial symmetry is retained. The exponent m is in fact to be determined (in virtue of the fact that we seek the minimum critical Rayleigh number) by the requirement that our asymptotic dependence of R_c upon T shall be the lowest possible power of T consistent with equation (3.3).

Now we have to investigate the possible transformations, of which there are relatively few, and find the one for which R_c varies as the lowest power of T. It goes almost without saying that R, which appears only in equation (2.2), must be present in some form in the limiting equation; otherwise the interior is described by the same equations as would be found in a treatment of the trivial problem of stability of uniform solid body rotation of a liquid. The instability must then originate in the boundary layer at the spherical surface, and the asymptotic limits imply that R_c varies as T. This is faster than the minimum rate of increase of R_c which in fact occurs when all three terms of equation (2.2) contribute.

If we take $\omega_0 = 0$, for the sake of argument, then the pertinent operators in (2.2)–(2.4) have the asymptotic expansions in the co-ordinates ω^*, z

$$\Delta_5 = T^{2m}\mathscr{D}_{\omega^*} + O(1), \quad \partial^2(1-\mu^2)/\partial\mu^2 = T^{2m}z^2\mathscr{D}_{\omega^*} + O(1), \tag{3.4}$$

where

$$\mathscr{D}_{\omega^*} \equiv \frac{1}{\omega^{*3}}\frac{\partial}{\partial\omega^*}\left(\omega^{*3}\frac{\partial}{\partial\omega^*}\right). \tag{3.5}$$

We can eliminate θ from the perturbation equations, whereby the asymptotic form of (2.2)–(2.4) becomes, for z bounded away from the spherical boundary,

$$T^{6m}\mathscr{D}^3_{\omega^*}U^* + T\partial^2 U^*/\partial z^2 = T^{2m}Rz^2\mathscr{D}_{\omega^*}U^* + O(T^{4m}), \tag{3.6}$$

$$\mathscr{D}_{\omega^*}V^* + \partial U^*/\partial z = O(T^{-2m}), \tag{3.7}$$

where

$$U \sim U^*, \quad V \sim V^* T^{-2m}. \tag{3.8}$$

In the limit $T \to \infty$, it follows easily from (3.6) that the slowest asymptotic increase of R is obtained when

$$m = \tfrac{1}{6}, \quad P \equiv R/T^{\frac{2}{3}} = O(1). \tag{3.9}$$

Equations (3.6) and (3.7) can then be reduced to the asymptotic form

$$\{\mathscr{D}^3_{\omega^*} + \partial^2/\partial z^2 - z^2 P\mathscr{D}_{\omega^*}\}\, U^* = O(T^{-\frac{1}{3}}), \tag{3.10}$$

$$\mathscr{D}_{\omega^*}V^* + \partial U^*/\partial z = O(T^{-\frac{1}{3}}). \tag{3.11}$$

Before we continue and seek a limit of (2.2)–(2.4) which will be valid near the spherical boundary, we point out that the solution (3.10) is of the form

$$U^* = \phi(z) J_1(\alpha\omega^*)/\omega^*, \tag{3.12}$$

where $J_1(\alpha\omega^*)$ is the bounded Bessel function of first order and first kind, α is a constant and $\phi(z)$ is an even solution of

$$\{(d^2/dz^2) + P\alpha^2 z^2 - \alpha^6\}\, \phi(z) = O(T^{-\frac{1}{3}}). \tag{3.13}$$

Now it follows, if we retain a non-zero value for ω_0, that we obtain equations similar to (3.10) and (3.11) where $\mathscr{D}_{\omega^*}$ is replaced by $\partial^2/\partial\omega^{*2}$. The corresponding dependence on ω^* then is as $\sin(\alpha\omega^* + \phi)$, and since sine is of $O(1)$ for all values of its argument we have to consider what happens when ω^* takes on values which correspond to values of ω near zero. In fact the solution obtained for $\omega_0 \neq 0$ must be matched to a solution of the form of equation (3.12) in order to satisfy the correct equation near the axis. It follows finally that bounded solutions obtained for $\omega_0 \neq 0$ have relative amplitudes which vanish as $T^{-\frac{1}{6}}$; hence we need not consider them.

It also follows from (3.12) that the perturbations which feed the boundary layer where $r \to 1$ will be proportional to

$$U(1, \mu) \propto J_1[\alpha T^{\frac{1}{6}}(1-\mu^2)^{\frac{1}{2}}]/(1-\mu^2)^{\frac{1}{2}} T^{\frac{1}{6}}. \tag{3.14}$$

Hence, we see that the boundary layer will be fed only in a region where $(1-\mu^2)^{\frac{1}{2}} = O(T^{-\frac{1}{6}})$. From (3.1) to (3.3) it then follows that the proper limit of (2.2)–(2.4) at the spherical boundary is for $z \to 1$ with ω^* fixed. This is equivalent to the observation that the incipient motions in the interior of the sphere are all centred in a cylindrical region of $O(T^{-\frac{1}{6}})$ around the axis of rotation and the boundary layer will be fed by motions only inside that region. In such a small region of the spherical surface, the fluid will encounter an almost plane boundary.

It should be pointed out here that Stewartson (1957) and Robinson (1959) found it necessary to consider other limits of the equations for small motions departing from rigid-body rotations. We have not had to consider these limits because our system is driven internally and the motions vanish at the boundaries;

theirs, on the other hand, were driven at the boundaries. In this problem the thermal driving term [which is proportional to R in (2.2)] has been shown to be of insignificant order throughout the sphere except at the axis of rotation. Thus it turns out that we are able to construct a solution which satisfies the boundary conditions by taking only two limits of the equations.

With the transformation (in the limit $T\to\infty$)

$$\tilde{z} = (1-z)\,T^{n}, \quad n>0, \tag{3.15}$$

we imply that, for $\tilde{z}$ fixed, $z\to 1$. The equation for $U^* \sim U$, under the transformation (3.3) and (3.15) obtained for $n=\frac{1}{4}$ with P and ω^* fixed, is

$$\{\partial^6/\partial\tilde{z}^6 + \partial^2/\partial\tilde{z}^2\}\,U^* = O(T^{-\frac{1}{4}}), \tag{3.16}$$

and for $V^* \sim T^{\frac{1}{4}}V$ the equation is

$$\partial^2 V^*/\partial\tilde{z}^2 - \partial U^*/\partial\tilde{z} = O(T^{-\frac{1}{4}}). \tag{3.17}$$

The solutions for U, V which will be valid asymptotic approximations to (2.2)–(2.4) for the entire range of z, [0, 1], are obtained by matching the bounded solutions of (3.10) and (3.11) to those of (3.16) and (3.17). The bounded solution of (3.10) has been obtained in (3.12) and (3.13), and the solution of (3.11) is calculated to be

$$V^*(z,\omega^*) = \phi'(z)\,J_1(\alpha\omega^*)/\alpha^{2*} + A(z). \tag{3.18}$$

The solution of (3.16) can be written as the real part of

$$\tilde{U}^*(\tilde{z},\omega^*) = \tilde{A}(\omega^*,T) + \tilde{B}(\omega^*,T)\exp(-\beta_1\tilde{z}) + \tilde{C}(\omega^*,T)\exp(-\beta_2\tilde{z}), \tag{3.19}$$

where β_1, β_2 are the square roots,

$$\beta_1 = \sqrt{i}, \quad \beta_2 = \sqrt{-i}, \tag{3.20}$$

that have positive real parts. The solution of (3.17) follows as the real part of

$$\tilde{V}^*(\tilde{z},\omega^*) = \tilde{D}(\omega^*,T) - \tilde{B}(\omega^*,T)\exp(-\beta_1\tilde{z})/\beta_1 - \tilde{C}(\omega^*,T)\exp(-\beta_2\tilde{z})/\beta_2. \tag{3.21}$$

We determine the function $\tilde{A}(\omega^*,T)$ from the requirement that the boundary layer should be fed by the interior, i.e.

$$\lim_{T\to\infty}\,[U^*(z,\omega^*) - \tilde{U}^*(z,\omega^*)] = O(T^{-\frac{1}{4}}). \tag{3.22}$$

It then follows from (3.12), (3.13) and (3.19) that

$$A(\omega^*,T) = J_1(\alpha\omega^*)\,\phi(1)/\omega^*. \tag{3.23}$$

The function $\tilde{D}(\omega^*,T)$ is similarly determined from the requirement

$$\lim_{T\to\infty}\,[\tilde{V}^*(z,\omega^*)\,T^{\frac{1}{4}} - V^*(\tilde{z},\omega^*)\,T^{\frac{1}{3}}] = O(T^{-\frac{5}{12}}), \tag{3.24}$$

which results, with the aid of (3.18) and (3.21), in

$$\tilde{D}(\omega^*,T) = [\phi'(1)\,J_1(\alpha\omega^*)/\alpha^2\omega^* + A(1)]\,T^{-\frac{1}{12}}. \tag{3.25}$$

The combination of results (3.19), (3.23), (3.21) and (3.25) yields the asymptotic eigenfunctions, which are uniformly valid approximations to (2.2)–(2.4) over the entire range of z, or r, in the small cylinder of $O(T^{-\frac{1}{6}})$ about the axis of rotation. These can be written as real parts of

$$U(z, \omega^*) = \phi(z)\, J_1(\alpha\omega^*)/\omega^* \\ + [\tilde{B}(\omega^*, T) \exp(-\beta_1 \tilde{z}) + \tilde{C}(\omega^*, T) \exp(-\beta_2 \tilde{z})]\,[1 + O(T^{-\frac{1}{6}})] + O(T^{-\frac{1}{6}}), \quad (3.26)$$

and

$$V(z, \omega^*) = \phi'(z)\, J_1(\alpha\omega^*)/\alpha^2\omega^* T^{\frac{1}{3}} + A(z)\, T^{\frac{1}{3}} \\ - [\tilde{B}(\omega^*, T) \exp(-\beta_1 \tilde{z})/T^{\frac{1}{4}}\beta_1 + \tilde{C}(\omega^*, T) \exp(-\beta_2 \tilde{z})/T^{\frac{1}{4}}\beta_2] \\ \times [1 + O(T^{-\frac{1}{6}})] + O(T^{-\frac{5}{12}}). \quad (3.27)$$

The functions $A(z)$, $\tilde{B}(\omega^*, T)$, $\tilde{C}(\omega^*, T)$ are determined from boundary conditions at $\tilde{z} = 0$.

4. The boundary conditions and characteristic equation

To obtain the characteristic equation for either the free or rigid boundary, (2.7) or (2.8) must be satisfied at $r = 1$ for all values of μ. It has been shown in the previous section that the asymptotic eigenfunctions which 'feed' the boundary layer, together with all their derivatives, are small ($O(T^{-\frac{1}{4}})$) everywhere except in a narrow cylindrical region near the axis of rotation of the sphere. In the limit of large Taylor number, it is hence sufficient, to $O(T^{-\frac{1}{4}})$, to have the boundary conditions satisfied on the circular cap at $z = 1$, for finite ω^*.

We shall omit some algebra and state the results: The characteristic equation for the free boundary is

$$\phi(1) = O(T^{-\frac{1}{6}}), \quad (4.1)$$

and the eigenfunction can be written, for $z > 0$, as

$$U(z_1 \omega^*) \sim \{\phi(z)\, J_1(\alpha\omega^*)/\omega^* \\ - 2\phi'(1)\, T^{-\frac{1}{2}} \exp(-\tilde{z}/\sqrt{2}) \sin(\tilde{z}/\sqrt{2})\, \partial(J_1(\alpha\omega^*))/\partial\omega^*\}\{1 + O(T^{-\frac{1}{6}})\}. \quad (4.2)$$

For the rigid boundary the characteristic equation is

$$\phi(1) = \phi'(1)/\alpha^2 \sqrt{2} T^{\frac{1}{12}}, \quad (4.3)$$

and the eigenfunction can be written, for $z > 0$, as

$$U(z_1 \omega^*) = \{\phi(z) - \phi'(1) \exp(-\tilde{z}/\sqrt{2}) \cos(\tilde{z}/\sqrt{2} - \tfrac{1}{4}\pi)/\alpha^2 T^{\frac{1}{12}}\} \\ \times J_1(\alpha\omega^*)/\omega^* + O(T^{-\frac{1}{6}}). \quad (4.4)$$

Similar expressions are obtained when $z < 0$.

We point out that, to describe the incipient motion for the case of free boundaries, we do not need a boundary layer; however, to compute the stresses at the free boundary, the second term in (4.2) must be retained. For a rigid boundary we must solve for $\phi(z)$ to $O(T^{-\frac{1}{12}})$ to compute the incipient motion and it is necessary to retain the second term in (4.3) for the boundary-layer motion as well as the stresses at the boundary.

We shall discuss a numerical solution of the characteristic equations (4.1) and (4.3) in § 6.

5. Velocity field for the rigid boundary

In this section we shall discuss the streamline pattern for the rotating sphere at the onset of steady convection in the case of rigid boundaries. For this purpose, it is convenient to write the dimensionless velocity components in the cylindrical co-ordinates

$$\mathbf{u} = -\omega \frac{\partial}{\partial z} U \hat{\boldsymbol{\omega}} + \omega T^{\frac{1}{3}} V \hat{\boldsymbol{\phi}} + \omega^{-1} \frac{\partial}{\partial \omega}(\omega^2 U)\,\hat{\mathbf{z}}. \tag{5.1}$$

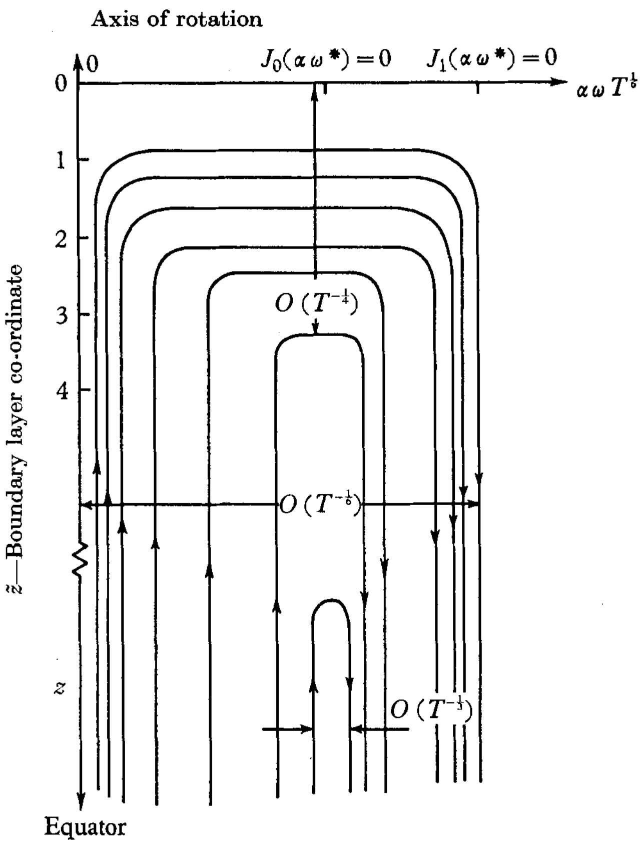

FIGURE 1. Schematic representation of the projections of the streamlines on a meridional plane for one-half of the toroidal cell closest to the axis of rotation. The fluid enters and leaves the boundary layer in regions $O(T^{-\frac{1}{4}})$ thick and the cell cores are $O(T^{-\frac{1}{3}})$ wide. The cell widths are $O(T^{-\frac{1}{6}})$. This cellular motion will decay as $T^{-\frac{1}{4}}$ with increasing ω.

From the results of §§ 3 and 4, the asymptotic form of the streamfunctions, U and V, can be written as

$$U = \{\phi(z) - G(\tilde{z})/T^{\frac{1}{12}}\}\, J_1(\alpha\omega^*)/\omega^* + O(T^{-\frac{1}{6}}), \tag{5.2}$$

$$T^{\frac{1}{3}} V = \{\phi'(z) - H(\tilde{z})\}\, J_1(\alpha\omega^*)/(2\alpha^2)^{\frac{1}{2}}\omega^* + O(T^{-\frac{1}{12}}), \tag{5.3}$$

where

$$\left.\begin{aligned} G(\tilde{z}) &= \phi'(1)\exp(-\beta_1\tilde{z})\cos(\tilde{\beta}_1\tilde{z} - \tfrac{1}{4}\pi)/\alpha^2, \\ H(\tilde{z}) &= \phi'(1)\exp(-\tilde{\beta}_1\tilde{z})\cos(\tilde{\beta}_1\tilde{z}), \\ \tilde{\beta}_1 = 1/\sqrt{2}, &\quad \tilde{z} = (1-z)\,T^{\frac{1}{4}}, \quad \omega^* = T^{\frac{1}{6}}\omega. \end{aligned}\right\} \tag{5.4}$$

For $(1-z) = O(1)$, which represents the interior of the sphere, the expression (5.1) reduces with the aid of (5.2) and (5.3) to

$$\mathbf{u} = -T^{-\frac{1}{6}}\phi'(z)\,J_1(\alpha\omega^*)\,\hat{\boldsymbol{\omega}} + \phi'(z)\,J_1(\alpha\omega^*)\,\alpha^{-2}\hat{\boldsymbol{\phi}} + \phi(z)\,\alpha J_0(\alpha\omega^*)\,\hat{\mathbf{z}}. \tag{5.5}$$

It now follows that, for large T, the interior motion consists of particle paths which are tightly wound spirals about the axis of rotation. There is a slow drift, of $O(T^{-\frac{1}{6}})$, of fluid normal to the axis, and it is this drift which induces the spiralling motion.

Since the motion in the interior of the sphere consists of fluid particles moving in spirals, the particles at the solid boundary are slowed down by Ekman layers. The terms $G(\tilde{z})$, $H(\tilde{z})$ in (5.2) and (5.3) represent this effect. Where the boundary layer co-ordinate $\tilde{z} = O(1)$, (5.1) can be written with the aid of (5.2), (5.3) and (4.15) as

$$\begin{aligned}\mathbf{u}(\tilde{z},\omega^*) \sim \phi'(1)\{&\exp(\tilde{\beta}_1\tilde{z})\sin(\tilde{\beta}_1\tilde{z})\,J_1(\alpha\omega^*)\,\hat{\boldsymbol{\omega}}\\ &+[1-\exp(-\tilde{\beta}_1\tilde{z})\cos(\tilde{\beta}_1\tilde{z})]\,J_1(\alpha\omega^*)\,\hat{\boldsymbol{\phi}}\\ &+T^{-\frac{1}{12}}\alpha[1-\exp(-\tilde{\beta}_1\tilde{z})(\cos\tilde{\beta}_1\tilde{z}+\sin\tilde{\beta}_1\tilde{z})]\,J_0(\alpha\omega^*)\,\hat{\mathbf{z}}\}/\sqrt{2}\alpha.\end{aligned} \tag{5.6}$$

A schematic representation of the streamline pattern for the toroidal component of (5.1) for one-half of the cell nearest to the axis of rotation is given in figure 1.

6. Some numerical calculations

In this section we shall consider a variational procedure for calculating the asymptotic form of the characteristic equation $P = P(\alpha)$. From the results of §4, (4.1) or (4.3) reduces in either case to

$$\phi(1) = O(T^{-\frac{1}{12}}). \tag{6.1}$$

We multiply (3.13) by $\phi(z)$ and integrate from $0 \to 1$, which yields, with the aid of (6.1), a variational formula for P as

$$\alpha^2 P = \int_0^1 \left[\left(\frac{d\phi}{dz}\right)^2 + \alpha^6\phi^2\right] dz \Big/ \int_0^1 (z\phi)^2\,dz + O(T^{-\frac{1}{12}}). \tag{6.2}$$

To obtain an upper bound for the minimum value of $\alpha^2 P$, we can substitute a trial function for $\phi(z)$ in (6.2). We expand $\phi(z)$ in a Fourier cosine series,

$$\phi(z) = \sum_{n=0}^{\infty} A_n \cos[(n+\tfrac{1}{2})\pi z], \tag{6.3}$$

which satisfies the boundary condition (6.1) to $O(T^{-\frac{1}{12}})$. With the substitution of (6.3) into the variational formula (6.2), the best approximation to $\phi(z)$ is provided by the choice of A_n which makes $\partial(\alpha^2 P)/\partial A_n = 0$:

$$\begin{aligned}\sum_{m=0}^{\infty} A_m \Bigg[\delta_{mn}\left\{\alpha^2 P\left(\frac{1}{3} - \frac{2}{(2n+1)^2\pi^2}\right) - \alpha^6 - (n+\tfrac{1}{2})^2\pi^2\right\}&\\ + (1-\delta_{mn})\,\alpha^2 P(-1)^{m-n}\frac{(2m+1)(2n+1)}{(m+n+1)^2(m-n)^2\pi^2}\Bigg] &= 0.\end{aligned} \tag{6.4}$$

From (6.4) we obtain the characteristic equation of infinite degree in $\alpha^2 P$ as

$$\det \left| \delta_{mn} \left\{ \alpha^2 P \left(\frac{1}{3} - \frac{2}{(2n+1)^2 \pi^2} \right) - \alpha^6 - (n+\tfrac{1}{2})^2 \pi^2 \right\} + (1-\delta_{mn})\, \alpha^2 P (-1)^{m-n} \frac{(2m+1)(2n+1)}{(m+n+1)^2 (m-n)^2 \pi^2} \right| = 0. \quad (6.5)$$

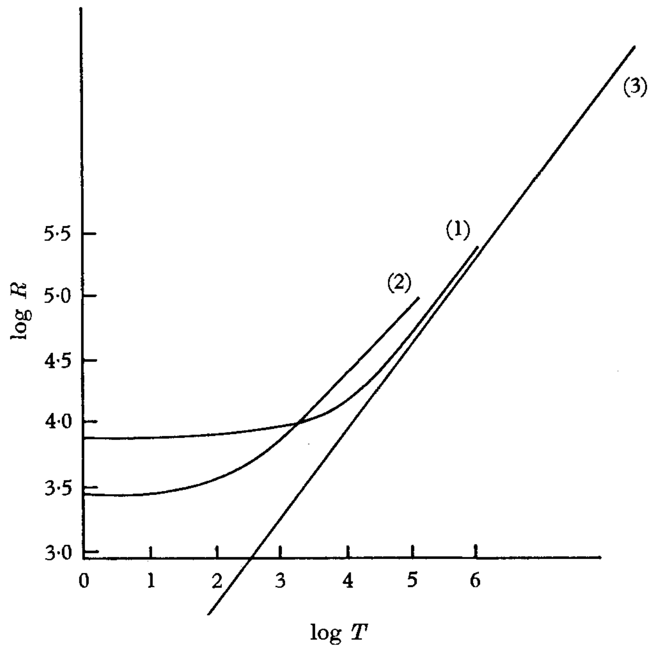

FIGURE 2. The dependence of the critical Rayleigh number for the onset of convective instability on the Taylor number in a rotating fluid sphere. The curves labelled (1) and (2) are previous calculations from Bisshopp (1958) for the cases of a rigid and free bounding surface. Curve (3) is the calculation of the asymptotic limit of curves (1) and (2).

By setting the corner element equal to zero, we obtain the characteristic equation

$$\alpha^2 P_{(1)} (\pi^2 - 6)/3\pi^2 = \alpha^6 + \tfrac{1}{4}\pi^2, \quad (6.6)$$

from which the minimum value $P_{c(1)}$ is calculated as

$$P_{c(1)} = \tfrac{9}{4}\pi^{\frac{10}{3}}/(\pi^2 - 6) \simeq 26{\cdot}42, \quad (6.7)$$

$$\alpha^2_{c(1)} = \tfrac{1}{2}\pi^{\frac{2}{3}} \simeq 1{\cdot}03. \quad (6.8)$$

Setting a 2 by 2 determinant equal to zero yields the results

$$P_{c(2)} \simeq 24{\cdot}47, \quad (6.9)$$

$$\alpha_{c(2)} \simeq 1{\cdot}07, \quad (6.10)$$

and a 3 by 3 determinant yields the results

$$P_{c(3)} \simeq 24{\cdot}33, \quad (6.11)$$

$$\alpha_{c(3)} \simeq 1{\cdot}09. \quad (6.12)$$

From the above calculations, it is suggested that a larger determinant would yield another 1 % accuracy, and it seems hardly worth while to carry them further. The result (6.11), valid to $O(T^{-\frac{1}{12}})$ for both boundary conditions, agrees well with the numerical calculations performed by Bisshopp (1958). This result, along with the previous result, is plotted on figure 2.

7. Concluding remarks

It is perhaps worth pointing out that the asymptotic analysis of the thermal instability in a sphere applies just as well for a rotating oblate spheroid with a uniform distribution of internal heat source. The basis of this observation is that the potentially unstable temperature gradient and gravity vectors in the small region around the axis of rotation are identical with those in a sphere. It is this temperature gradient in combination with the gravity force which determines the asymptotic critical Rayleigh number at the limit of large Taylor number.

A second point which must be mentioned is that the assumptions of the exchange of stabilities and axial symmetry made here are unjustifiable. It is reasonable to expect that for large Prandtl number only will the instability set in as steady convection rather than overstability, as is the case for the rotating plane layer (cf. Chandrasekhar 1961, ch. 3). As we have seen, asymptotic approximation provides a powerful method for dealing with axisymmetric, neutral modes of thermal instability in rapidly rotating systems. It is hoped that it will be possible also to consider aspects of the non-axisymmetric problem and/or overstable modes with such methods.

Since the completion of this work, P. H. Roberts (1965) has communicated to us the results of his independent formulation of the problem. He computed values of P_c and α_c by direct numerical integration of equation (3.13), which he also derives. His value of P_c turns out to be a puzzling 20 % lower than ours.

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